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Unveiling Mathematical Elegance: Exploring The Synergy Between Algebra and Geometry

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ABSTRACT

In today's dynamic and complex society, the ability to represent mathematical concepts and solve real-world problems is essential for mathematics education students. This study explores how algebraic and geometric representations can be synergized to solve environmental problems commonly faced in everyday life, particularly through the "builder's problem" involving quadrilaterals. Using a qualitative, exploratory case study approach, one student with high mathematical ability was selected through a 10-item Mathematical Ability Test (TKM). Data collection included a problem-solving task and semi-structured interviews to examine the student's reasoning processes. Data were analyzed using Miles and Huberman's model: data condensation, data display, and conclusion drawing. Triangulation of test results, written work, and interview data ensured the validity of findings. Results indicate that the student was able to construct a mathematical model of the problem algebraically and then reinterpret it geometrically, revealing an equivalent but more conceptually elegant solution. Specifically, solving for the yard boundary of a rectangular plot algebraically led to the same value as determining the radius of an incircle within a right triangle geometrically. However, while the algebraic representation was more precise, the geometric approach offered deeper visual insight into the structure of the problem. This study highlights the importance of dual representations in enhancing mathematical understanding and problem-solving. It suggests the integration of real-world contexts and multiple solution strategies in teacher education programs, and recommends further research with diverse participants to deepen insights into algebra-geometry interactions in problem solving.

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1. INTRODUCTION

Mathematics serves as a foundational science that underpins the advancement of science, technology, and innovation (Pinto & Cañadas, 2021). Its central role across disciplines lies not only in its computational utility but also in its capacity to develop logical, abstract, and structured reasoning.

Lerman (2020) classifies mathematics into four principal domains: arithmetic, algebra, geometry, and analysis. Among these, algebra and geometry play distinct yet interrelated roles in representing and solving problems in both academic and real-world contexts.

Algebra is defined as a mathematical discipline that employs symbolic statements to express relationships and structures. It functions as a powerful tool for generalization and abstraction (Pinto & Cañadas, 2021). In addition to its foundational importance in pure mathematics, algebra is instrumental in solving problems in diverse fields such as science, economics, business, computing, and engineering (Denecke & Wismath, 2018; Grønmo, 2018; Juraev & Bozorov, 2024; Rojano & Palmas, 2022; Tung et al., 2024). Scholars argue that algebra is more than a computational skill—it is a form of mental activity involving pattern recognition, variable manipulation, and symbolic representation (Kaput et al., 2017; Kieran, 2020).

Geometry, on the other hand, enables individuals to interpret and understand the spatial world around them by analyzing shapes, forms, and their interrelations (Elia et al., 2018; Gambini & Lénárt, 2021). Hourigan and Leavy (2017) emphasize the importance of geometry in developing deductive reasoning and visual-spatial thinking. In this regard, geometry supports not only mathematical cognition but also broader cognitive development. Ani (2021) expands this view by positioning mathematics—especially through geometry—as a lens through which to understand societal challenges, not merely as a collection of rote formulas. He contends that mathematics should be viewed as a thinking tool, a window to reality, rather than a set of memorized rules.

These two branches—algebra and geometry—can be seen as complementary modes of reasoning. When applied in tandem, they offer multiple pathways to understanding and solving problems. As Simon (2017) explains, mathematics often manifests as an abstract endeavor that requires the mental coordination of complex structures and representations. This duality also connects with Renert's (2011) notion of mathematical elegance, where life mathematics and academic mathematics converge as explorations of logical beauty and structured reasoning.

One compelling way to explore the interplay between algebraic and geometric thinking is through real-world problems. A well-known example is the "builder's problem" (Zhang et al., 2024), which asks students to determine the dimensions or layout of a house and yard based on spatial and proportional constraints. This problem, while simple on the surface, can be approached through algebraic modeling or geometric reasoning—or ideally, both. As such, it becomes an ideal context for investigating how students connect mathematical representations to real-life situations.

Unfortunately, the literature has consistently shown that pre-service teachers and students often struggle to link geometric concepts accurately, particularly in the domain of quadrilaterals (Avcu, 2023; Brunheira & da Ponte, 2019; Okazaki & Fujita, 2007). Fujita and Jones (2014), in a study involving pre-service elementary teachers, reported misconceptions regarding the hierarchy of quadrilaterals. For example, many participants believed that a rectangle was a specific type of square—a conceptual inversion that persisted despite formal instruction. This indicates not only a lack of conceptual understanding but also a disconnect between theoretical knowledge and practical application.

Monaghan (2000) and Wu and Ma (2005) found similar issues in students' understanding of quadrilaterals. Although instructional materials often cover the six types of convex quadrilaterals—parallelogram, trapezoid, kite, rectangle, rhombus, and square—students tend to have difficulty applying these definitions in real-life or visual contexts (Clements & Sarama, 2011). These findings suggest that even well-taught content can remain abstract or disconnected unless contextualized meaningfully.

Thus, understanding the concept of quadrilaterals is critical for both teaching and learning geometry at all educational levels. As Al-Mutawah et al. (2019) and Chapman (2017) emphasize, conceptual understanding in mathematics requires more than factual knowledge—it requires the ability to interconnect concepts, recognize structures, and apply definitions flexibly. Chai et al. (2020) further differentiate this understanding into category knowledge (the ability to classify based on essential attributes) and classification knowledge (the ability to define and distinguish among categories).

In this study, mathematical ability is defined operationally as the student's capacity to analyze and solve mathematical problems involving both algebraic and geometric representations. Participant selection was conducted using the Mathematical Ability Test (TKM), which comprised 10 open-ended questions designed to assess a range of problem-solving skills. Based on the test results, students were categorized into three levels of ability: high ability (scores between 80 and 100), medium ability (scores between 60 and 79), and low ability (scores of 60 or below) (Ratumanan & Laurens, 2011).

This research focused on a single participant from the high-ability group, selected with the assumption that such a student would possess the necessary proficiency to engage in dual mathematical representations and navigate complex problem-solving tasks. The analysis of the participant's thinking and performance was guided by a four-stage problem-solving framework adapted from Hong and Kim (2016), Jonassen (1997), and Shin, Jonassen, and McGee (2003). The stages include *Problem Representation*, *Generating Solutions*, *Justification*, and *Monitoring and Evaluation*.

Each of these stages was examined through specific indicators. For instance, in the *Problem Representation* stage, attention was paid to how the student restated the problem in their own words and identified key terms. In the *Generating Solutions* stage, the focus shifted to how the student constructed mathematical models or visual representations. The *Justification* stage evaluated the logical reasoning and argumentation supporting the solution, while the *Monitoring and Evaluation* stage assessed the student's ability to reflect on and verify the accuracy and appropriateness of their solution.

By exploring how a student bridges algebra and geometry in solving the builder's problem, this study aims to reveal the cognitive and representational strategies that support deep mathematical understanding. Such insights are crucial for improving mathematics education programs that prepare future teachers to guide students through multifaceted problem-solving processes.

2. METHODS

This study employed an exploratory qualitative design aimed at uncovering in-depth insights into how students with high mathematical ability utilize both algebraic and geometric representations in solving real-world contextual problems—specifically, a builder's problem involving quadrilaterals. A case study approach was used, focusing on a single participant to allow for detailed analysis of cognitive and metacognitive processes during problem solving.

2.1 Research Participant

The research subject was one student from the Mathematics Education Study Program at FKIP Unismuh Makassar, selected based on their demonstrated high mathematical ability. The selection was made using a standardized instrument developed by the researchers, referred to as the Test of Mathematical Ability (TKM).

2.2 Instruments

Three primary data collection instruments were employed to capture different dimensions of the student's mathematical thinking:

1. Test of Mathematical Ability (TKM)

This instrument served as a screening tool to identify students with high mathematical ability. It consisted of 10 open-ended, descriptive items designed according to the Semester Learning Plan (RPS) of the Mathematics Education Program. The test assessed students' problem-solving processes, conceptual understanding, and ability to model mathematical relationships. Students who achieved a score in the range of 80 to 100 were classified as having high mathematical ability, following the criteria established by Ratumanan and Laurens (2011). Only students in this category were considered eligible for participation in the core study.

2. Problem Solving Task (TPM)

The main research instrument was a contextual builder's problem involving a rectangular land plot and its surrounding yard—integrating aspects of both algebra and geometry. This problem was carefully designed to elicit dual representations by requiring the student to determine the boundary of the yard based on a fixed area constraint. The task aimed to uncover the student's approach to modeling, interpreting, and reasoning using both symbolic (algebraic) and visual-spatial (geometric) strategies.

3. Semi-Structured Interview

Since not all cognitive processes and reasoning strategies are evident from written responses alone, semi-structured interviews were conducted to explore the participant's thought processes further. The interview protocol included both prepared questions and dynamic prompts tailored to the student's written responses and verbal explanations. Interviews were audio-recorded and supported by field notes capturing gestures, expressions, and mimics that provided additional insight into the student's reasoning.

2.3 Data Collection Procedure

Data collection took place in three stages:

- Stage 1: Administration of the TKM to a group of prospective participants. One student scoring within the high-ability category was selected.
- Stage 2: The selected student completed the TPM under researcher supervision in a controlled, distraction-free setting.
- Stage 3: A follow-up interview was conducted to probe the student's reasoning, including justifications for the steps taken and the rationale for switching between algebraic and geometric approaches.

2.4 Data Analysis Techniques

The data were analyzed using the interactive model of qualitative analysis proposed by Miles, Huberman, and Saldaña (2014), involving the following stages:

1. Data Condensation

This stage involved selecting, focusing, simplifying, and transforming the raw data. Specific activities included:

- Re-reading and coding the researcher's notes, test responses, and interview transcripts.
- Analyzing the student's oral responses along with non-verbal cues such as gestures and facial expressions during interviews.
- Identifying key themes and reducing redundant or irrelevant information to highlight meaningful patterns of thinking.

2. Data Display

Data were organized and displayed to facilitate interpretation. This involved:

- Triangulating findings from the TKM, the problem-solving task, and the interviews to ensure consistency and credibility of interpretations.
- Developing visual maps and narrative sequences to describe the participant's cognitive process across the four stages of problem solving: *problem representation*, *generating solutions*, *justification*, and *monitoring and evaluation*.
- Representing how the student transitioned between algebraic and geometric reasoning, and identifying which representations were used more dominantly or effectively in different phases of the solution.

3. Drawing Conclusions and Verification

Patterns and relationships in the data were examined to draw conclusions regarding the participant's use of dual representation in mathematical problem solving. These conclusions were then verified through member checking and cross-verification across the multiple data sources to ensure validity. Only data that demonstrated consistency across methods were considered as confirmed findings.

2.5 Ethical Considerations

The research was conducted with informed consent from the participant, ensuring confidentiality and the right to withdraw at any stage. All data were anonymized during transcription and analysis to protect the participant's identity.

3. FINDINGS AND DISCUSSION

JS is a student of Mathematics Education study program of Universitas Muhammadiyah Makassar. JS has high mathematical ability and is able to communicate well. The following describes the results of JS's work and the researcher's interview with JS based on the stages of problem solving through analytical or algebraic solutions offered to geometric interpretations and solutions.

3.1 Problem Representation

The builder's problem written by (Jupri et al., 2024) is an illustration of the practice of reexamining analytic solutions. The simple example and problem explanation in Figure 1.1 below are adapted from the article (Arcavi & Resnick, 2008) entitled "*Generating Problems Out of Problems and Solutions Out of Solutions.*"

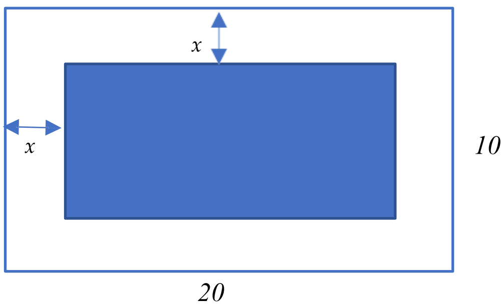

Figure 1. Problematics

The following is a description of Figure 1, namely "A builder will build a house on land measuring 20 *m* x 10 *m* according to the owner's request. The house to be built must have a yard around it and its area is equal to the area of the house building (see sketch Figure 1.1). Determine the size of the yard boundary around the house (x)!"

The subject began to read the problem presented by vocalizing, then drew a square as a form of interpretation related to the problem given. The subject presented a picture of a rectangle as shown in Figure 2.

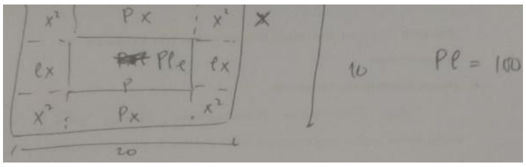

Figure 2. Problem Interpretation

Based on JS's written answers, it shows that the subject is able to represent the problem using his own language. In this case, JS used a pictorial representation, namely a rectangular image with length = 20 and width = 10. To explore keywords related to the problem given, the researcher conducted an interview as in the following interview excerpt.

P : Can you explain! How did you get this relationship? (pointing to figure 2)

JS : Think for a moment! The distance between the house and the perimeter of the yard is x, in this case I added an auxiliary line to get a square of four, I symbolize x^2. Then we get the area of the rectangle, which is twice Px and twice Lx

P : There's more?

JS : The area of PL is equal to one hundred.

P : The reason?

JS : Go back to reading the problem! The area of the house is equal to the area of the yard around it, so I conclude that each area is equal to half of the size of the land, which is half of twenty meters times ten meters.

Based on the interview transcript above, it shows that JS understood the problem given by determining three keywords that need attention in the problem solving process. The three keywords, namely (1) The size of the land is 20 m x 10 m; (2) The area of the house is equal to the area of the yard around it; (3) JS made a line to obtain the relationship in the form of a square $4x^2$, 2 Px, dan 2 Lx.

3.2 *Generating Solution*

JS constructed a solution by creating a mathematical pattern through the equation $P\ x\ L = 4x^2 + 2Px\ 2Lx$. In this case the building length (P) = 20 – 2x, width (L) = 10 – 2x, and the value of P + L = 30 – 4x. As shown in figure 1.3 below.

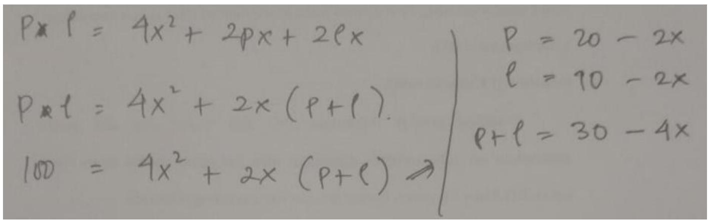

Figure 3. Generating Solution

Then it is emphasized through the following interview excerpt.

P : How do you get the mathematical equation? (pointing to figure 1.3)

JS : According to the previous assumption, namely $4x^2 + 2Px + 2Lx$ is the yard area while P by L is the house building area.

P : The reason?

JS : Because it is in accordance with the information contained in the problem, namely the building area of the house is the same as the surrounding yard area.

P : *Next step?*
JS : *Based on the equation, it can be simplified into* $4x^2 + 2x\ (P + L) = 100$ *while pointing to Figure 3.*

Based on the interview transcript, it was revealed that at the Generating Solution stage, JS built a solution by making a mathematical equation, namely $4x^2 + 2x\ (P + L) = 100$ and able to define the size of the house based on the written answer (Figure 1.3), namely $P = 20 - 2x$ and $L = 10 - 2x$ and obtain the result P + L = 30 – 4x.

3.3 *Justification*

The next stage, JS claimed or justified the solution through the quadratic equation formula. As shown in Figure 1.4 below.

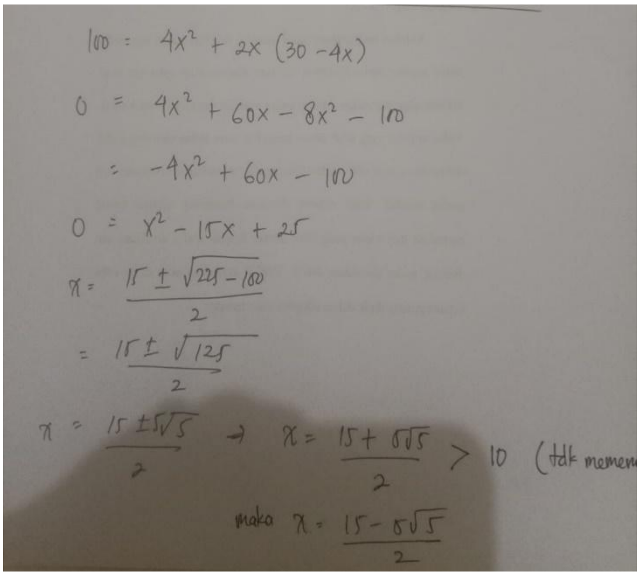

Figure 4. Justification

Then explained in the interview as follows.

P : *Can you explain! How did you get the value* $x = \frac{15-5\sqrt{5}}{2}$*? (pointing to figure 1.4)*
JS : *The first step is to form the equation* $4x^2 + 2x\ (P + L) = 100$*). Value 30 – 4x substituted into (P + L) to get the equation form* $4x^2 + 2x\ (30 - 4x) = 100$ *and obtained the simple form* $0 = x^2 - 15x + 25$.
P : *Next step?*
JS : *To get the x value, I use the ABC formula.*
P : *Yes! Why does the value* $x = \frac{15 + 5\sqrt{5}}{2}$ *not satisfy?*
JS : *Because the value of x is longer than the known land size in the problem, i.e. x > 10.*

Based on the interview transcript, it was revealed that at the Justification stage, JS was able to determine the value of $x = \frac{15-5\sqrt{5}}{2}$ as a solution or alternative to solving the problem given.

3.4 *Monitoring and Evaluation*

JS monitors and evaluates the results of his work based on his justifications as clarified in the following interview excerpt.

P : *Please reread the steps of the solution you obtained! What guarantees that the building area* *of the house is equal to 100 square meters?*

JS : *According to the information contained in the problem, the building area of the house is equal to the area of the yard around it or* $\frac{1}{2}(20\,.\,10) = 100$

P : *The next step is how to determine the boundary between the house and the yard based on* $\frac{1}{2}(20\,.\,10) = 100$*?*

JS : *May I write sir?*

P : *Please!*

JS : *Start writing*

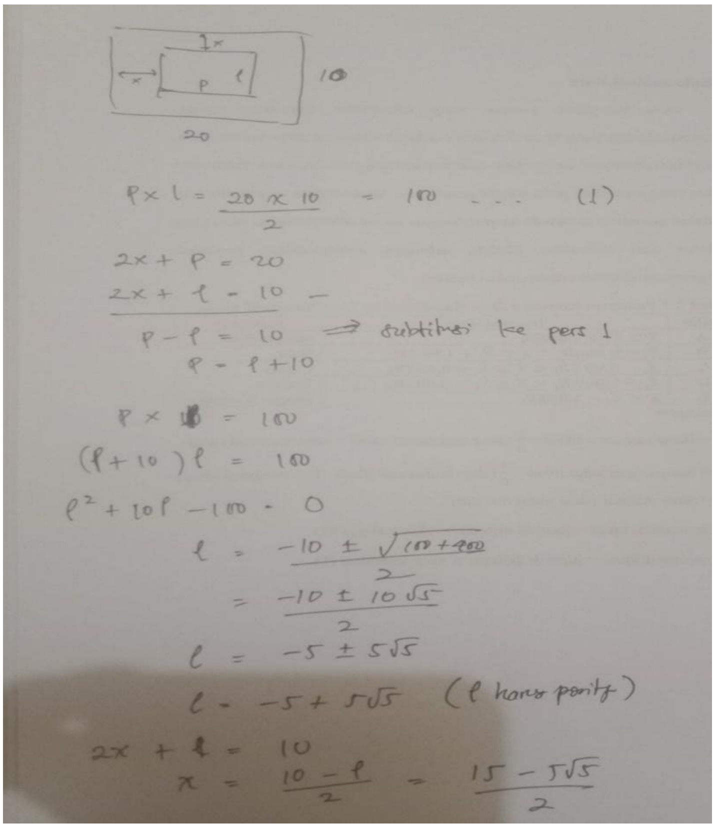

Figure 4. Monitoring and Evaluation

Here are the results that I obtained sir (while showing the answer sheet)!

P : *Yes! Can you explain how to get the same result?*

JS : *For example, if P x L is the area of the house (while pointing to the rectangle) then the area is* $\frac{20\ x\ 10}{2} = 100$ *as equation (1). The next step, which is 2x + P = 20 and 2x + L = 10 according to the size of the land. So to get the value of P = L + 10, then it can be taken by the difference between 2x + P = 20 with 2x + L = 10.*

P : *Next How?*

JS : *I substituted the value of P = L + 10 into equation 1, which is (P x L) = 100 obtained (L + 10) x L = 100 and obtained the quadratic equation* $L^2 + 10L - 100 = 0$*.*

P : *ow do you claim that a positive L value is satisfied?*

JS : *Silence for a moment! Because based on the form 2x + L = 10 and substituting the value of L, the same result is obtained, namely* $x = \frac{15-5\sqrt{5}}{2}$*.*

From the interview excerpt above, it is revealed that at the Monitoring and Evaluation stage JS was able to provide logical reasoning, namely by substituting the value of P = L + 10 into the equation P x L = 100 to obtain the quadratic equation $L^2 + 10L - 100 = 0$. JS concluded that by substituting the positive value of L, namely $L = -5 + 5\sqrt{5}$ into the form of the equation 2x + L = 10 the same result is obtained, namely $x = \frac{15-5\sqrt{5}}{2}$.

Based on the explanation above, it is indicated that JS conducted a logical thinking process to solve the builder's problem through analytical or algebraic representations. Furthermore, the geometric representation is described as follows.

3.5 Problem Representation

The subject re-read the problem presented, then drew a rectangle as a form of interpretation in geometry related to the problem given. The subject presented a picture of a rectangle as shown in the following figure 1.5.

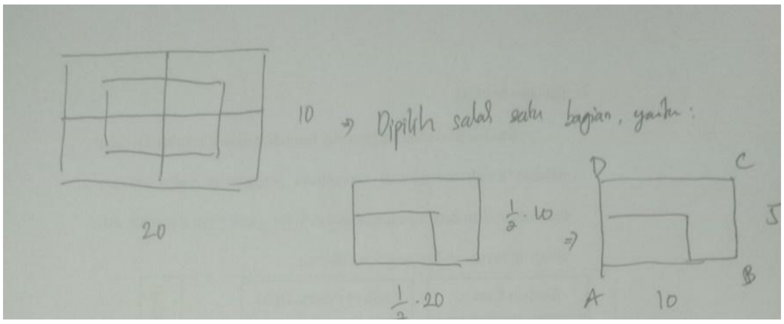

Figure 5. Problem Interpretation

Based on JS's written answers, it shows that the subject is able to represent the problem geometrically. In this case, JS used a pictorial representation, namely a rectangular image with length = 20 and width = 10. To explore keywords related to the problem given, the researcher conducted an interview as in the following interview excerpt.

P : Can you explain! How did you get this relationship? (pointing to figure 1.5)
JS : I divided the image in the problem into four parts, to obtain four congruent rectangles. So that each part has a length equal to 10 and a width equal to 5.
P : The reason?
JS : Just a guess sir.

Based on the interview transcript above, it shows that JS did not understand the problem given. In this case, JS was unable to provide logical reasoning and was only based on assumptions or conjectures by determining two keywords that need attention in the problem solving process. The two keywords, namely (1) dividing a rectangular shape into four parts (2) four congruent rectangles.

3.6 Generating Solution

JS constructs a solution through the congruence of two triangles to obtain the relationship $\overline{DT} = \overline{BU} = \overline{OP}$ if and only if ΔBUV is congruent to ΔVOP and ΔDTS is congruent to ΔOPS as shown in Figure 1.6 below.

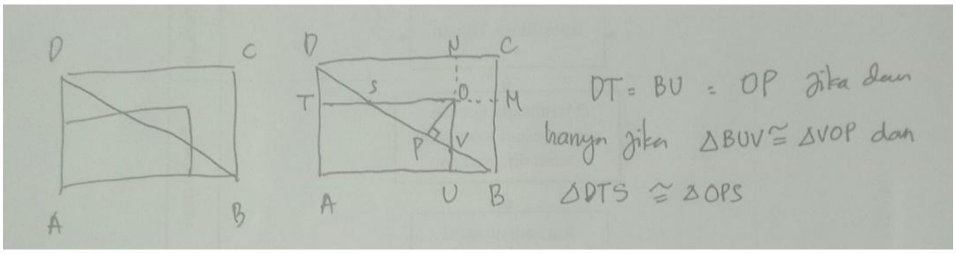

Figure 6. Generating Solution

Then it is emphasized through the following interview excerpt.

P : What guarantees that triangle BUV is congruent to triangle VOP and triangle DTS is congruent to triangle OPS? (pointing to figure 1.6)

JS : Pause for a moment! For triangle BUV, it is congruent to triangle VOP because angle V in triangle BUV is equal to angle V in triangle VOP because the angles are opposite and angle U in triangle BUV is equal to angle P in triangle VOP.

P : What about the angle pair B in triangle BUV and angle O in triangle VOP?

JS : Shut up.

Based on the interview transcript, it was revealed that at the Generating Solution stage, JS constructed a solution through the congruence of two triangles, although JS was unable to provide reasons related to the question of why the two triangles are congruent. In this case, based on the known pairs of angles that are congruent and JS was only able to reveal two pairs of angles that are congruent.

3.7 Justification

The next stage, JS claimed or justified the solution through the inner circle of a triangle of radius *r*. As shown in Figure 1.7 below.

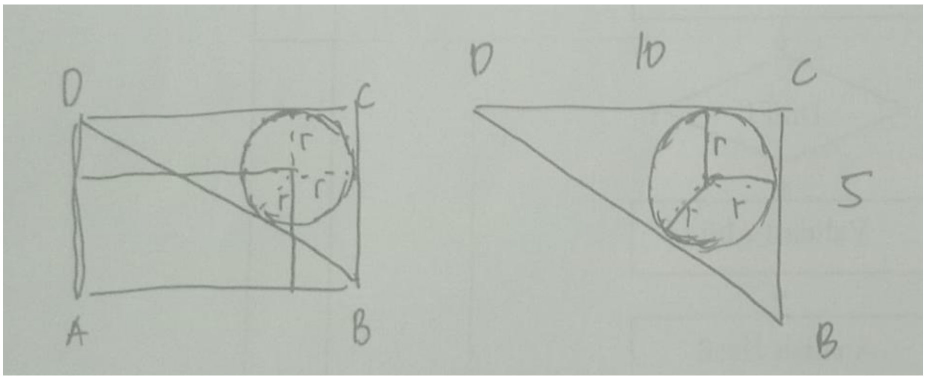

Figure 7. Justification

Then explained in the interview as follows.

P : How do you derive the relationship of the circles in a triangle (while pointing to Figure 1.7)?

JS : Silence for a moment! Based on the congruence of two triangles, we get the side length $\overline{DT} = \overline{BU} = \overline{OP}$ which is the radius of the inner circle of the triangle.

P : Are there any alternatives on how to obtain the inner circle of a triangle

JS : Silence for a moment! According to the lecture material that I have learned, namely using the angle bisector.

P : Okay! Can you mention the angle bisector postulate?

JS : An angle bisector is a line that divides an angle into two equal parts.

Based on the interview transcript, it was revealed that at the Justification stage, JS was able to define the postulate of the line dividing the angle, which is the line that divides the angle into two equal parts and was able to find the relationship of the circle in a triangle can be known through the congruence of two triangles and the postulate of the line dividing the angle.

3.8 Monitoring and Evaluation

JS monitors and evaluates the results of his work based on his justifications as shown in Figure 1.8 below.

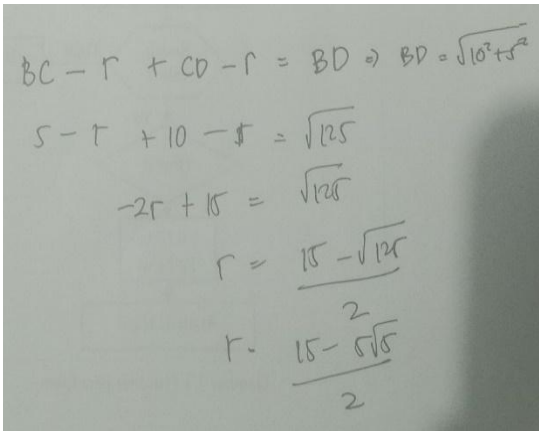

Figure 8. Monitoring and Evaluation

Then explained in the interview as follows.

P : Can you explain how you obtained the value $r = \frac{15-5\sqrt{5}}{2}$*?*
JS : First, I define the length of BD (while pointing to Figure 1.7) as the hypotenuse of a right triangle or the length of BC – r + CD – r = BD.
P : Why?
JS : Because the definition of angle bisector, which is the distance of the bisector to the legs of the angles of a triangle is equal in length or congruent.
P : Okay! What's the next step?
JS : The value of r can already be determined through the equation, which is $5 - r + 10 - r = \sqrt{125}$ *or* $r = \frac{15-5\sqrt{5}}{2}$

From the interview excerpt above, it was revealed that at the Monitoring and Evaluation stage JS was able to provide logical reasoning, namely to obtain the value of $r = \frac{15-5\sqrt{5}}{2}$ can be known through the equation BC – r + CD – r = BD or $5 - r + 10 - r = \sqrt{125}$.

Based on the explanation above, it is indicated that JS did the thinking process in solving the builder's problem through geometric representation. Although there are steps of completion that JS has not been able to reveal the reasons for.

Discussion

Based on data analysis, it is indicated that the subject has carried out the thinking process in solving the builder's problem through algebraic representation to geometric representation through four stages, namely Problem Representation, Generating Solution, Justification, and Monitoring and

Evaluation. However, in the subject's thinking process in solving the builder's problem, there are some unique activities and need to be studied more deeply as research findings. In this case, we will discuss the similarities and differences in algebraic representation and geometric representation of mathematics education student subjects in solving the builder's problem as follows.

Problem Representation

The subject represented the problem through his own language. In the form of a rectangular image, as a form of interpretation, algebraically and geometrically related to the problem given. This is in accordance with the research results of Chi & Glaser (1985); Cho & Kim (2020) revealed that problem solving is a specific context, making learning more interesting and meaningful for learners (read students), and encourages learners to define their own problems and determine the information and techniques needed to solve problems. The expression is reinforced by Facione & Facione (2007) opinion, describing that at the interpretation stage, students understand the problem by expressing the meaning or meaning of various experiences, situations, data, events, assessments/opinions, rules/probables, beliefs, rules, procedures, or criteria.

Generating Solution

The subject constructed the solution by making a mathematical equation, namely $4x^2 + 2x\,(P + L) = 100$ and was able to define the size of the house based on the written answer (Figure 1.3), namely $P = 20 - 2x$ dan $L = 10 - 2x$ and obtained the result P + L = 30 – 4x. Geometrically, the subject constructed the solution through the congruence of two triangles to obtain the relationship $\overline{DT} = \overline{BU} = \overline{OP}$ if and only if ΔBUV is congruent to ΔVOP and ΔDTS is congruent to ΔOPS. Although the subject seemed to have difficulty giving the right reason related to the question of why the two triangles are congruent. In this case, based on the known pairs of angles that are congruent and the subject is only able to reveal two pairs of angles that are congruent. This is in accordance with the theory put forward by Jonassen (1997) "Generate possible problem solutions," which means that learners (read students) try to generate possible solutions to the given problem. Then strengthened by the results of research by Cho & Kim (2020) which suggests that at the create stage, students make various plans to solve the problem.

Justification

The subject claims or justifies the solution through the quadratic equation formula and determines the value of $x = \frac{15-5\sqrt{5}}{2}$ as a solution or alternative solution to the problem given algebraically. While geometrically, claiming or justifying the solution through the inner circle of a triangle of radius r and being able to define the postulate of the angle bisector, which is the line that divides the angle into two equal parts and is able to find the relationship of the circle in a triangle can be known through the congruence of two triangles through the postulate of the bisector. This is in accordance with The problem offered Arcavi & Resnick (2008) can be solved algebraically, for example, the area of the yard is equal to the area of the house building, which is half of the total land area, so an equation is obtained:

$$(20 - 2x)\,.\,(10 - 2x) = \frac{1}{2}\,.\,20\,.\,10$$

The equation is a quadratic equation that can be solved algebraically, as follows.

$$200 - 40x - 20x + 4x^2 = 100$$
$$4x^2 - 60x + 200 = 100$$
$$4x^2 - 60x + 100 = 0$$

$$x^2 - 15x + 25 = 0$$
$$x_{1,2} = \frac{15 \pm \sqrt{225 - 100}}{2}$$
$$x_{1,2} = \frac{15 \pm \sqrt{125}}{2}$$

Possible values of x according to the problem conditions are

$$x = \frac{15 - 5\sqrt{5}}{2} \approx 1{,}9$$

Thus, the boundary of the house yard against the perimeter of the lot size is 1.9 meters.

Furthermore, related to the core of the builder's problem above, information was obtained that "The size of the yard boundary is equal to a quarter of the difference between the sum of the lengths of the two different sides of the lot size and the length of its diagonal (algebraically)." This means that the problem is the same as determining the length of the radius of the inner circle of a triangle (geometrically) obtained through the concept of the bisector of a triangle. The next question is why is this finding so important? This is because the definition of bisector of an angle revealed by the subject is incorrect, namely *"the bisector of an angle is the ray that separates the given angle into two congruent angles"* (Shen, 2016). If explored again from the definition of an angle "two rays with the same base," then the definition of a line for an angle becomes inappropriate, especially on the word "separate," i.e. an angle divided (separate), is it still an angle? Due to the weakness of this definition, the authors "propose" a definition of bisector/angle bisector such as "an angle bisector is a line that passes through the vertex of an angle and the interior of that angle, so that two congruent adjacent angles are formed" (Alexander & Koeberlein, 1999).

Monitoring and Evaluation

The subject carried out the thought process of solving the builder's problem through analytical or algebraic representation, in this case the subject was able to reveal and provide logical reasons, namely by substituting the value P = L + 10 into the equation P x L = 100 to obtain the quadratic equation $L^2 + 10L - 100 = 0$ The subject concluded that by substituting the positive value of L, namely $L = -5 + 5\sqrt{5}$ into the equation form 2x + L = 10 the same result is obtained, namely $x = \frac{15-5\sqrt{5}}{2}$ This is in accordance with the theory put forward by Jonassen (1997) *"Monitor the problem space and solution options,"* which means that the learner (read student) looks back between the problem and solution options. Then strengthened by the results of research by Cho & Kim (2020) which suggests that at the Evaluate stage, students assess the problem solving process which becomes the final solution. While geometrically interpreting, the subject carried out the thinking process in solving the builder's problem even though there were steps of completion that the subject had not been able to reveal the reasons for, so the researcher provided metacognitive scaffolding in helping to develop solutions to solve problems and strategic scaffolding to help identify information and make good use of it to discuss the suitability of the final solution chosen. This is in accordance with the findings of Ge & Land (2003); Jonassen (1997) who said that the stages of monitoring and justifying solutions are needed to solve problems. As reported in studies (Araiku et al., 2019; Chen & Bradshaw, 2007; Davis, 2000; Ge et al., 2005; Ge & Land, 2003; Greene & Land, 2000; Jonassen, 1997; Kim et al., 2015; Lee et al., 2014; Xun & Land, 2004), scaffolding is effective in problem solving and qualitatively improves it. Furthermore, the fact that facilitating problem solving with metacognitive scaffolding meant that the scaffolding provided in this study was dependent on the problem-solving circumstances. It is revealed the fact that scaffolding is provided according to the subject's circumstances which means that with the help of scaffolding the subject can do what he cannot do on his own.

4. CONCLUSIONS

The findings of this study reveal that the essence of the builder's problem lies not merely in determining the boundary dimensions of a plot of land, but in uncovering a deeper mathematical relationship—specifically, the equivalence between that boundary and the radius of an incircle in a right triangle. This insight highlights the potential of real-world problems to foster dual mathematical representations, integrating both algebraic and geometric reasoning. The study contributes to mathematics education by demonstrating how contextual problems can reveal elegant and meaningful connections between different mathematical domains. However, a key limitation of this research is its narrow scope, as it focuses on a single high-ability student, which restricts the generalizability of the findings. Future research is encouraged to expand this exploration by involving a broader and more diverse participant pool, including students of varying ability levels. Additionally, comparative studies examining how different learners approach similar problems across multiple contexts could further enrich our understanding of representational thinking in mathematics. Researchers are also encouraged to develop and test similar contextual tasks in varied instructional settings to explore their pedagogical implications more broadly.

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